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A RAMSEY THEOREM FOR TREES, WITH AN APPLICATION TO BANACH SPACES

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ABSTRACT

Let S be the binary tree of all sequences of 0's and 1's. A chain of S is any infinite linearly ordered subset. Let $\mathscr{X}$ be an analytic set of chains, we show that there exists a binary subtree S' of S such that either all chains of S' lie in $\mathscr{X}$ or no chain of S' lies in $\mathscr{X}$. As an application, we prove the following result on Banach spaces: If $(x_s)_{s\in S}$ is a bounded sequence of elements in a Banach space E, there exists a subtree S' of S such that for any chain β of S' the sequence $(x_s)_{s\in\beta}$ is either a weak Cauchy sequence or equivalent to the usual l^1 basis.

§0. Introduction

Throughout the paper S stands for the set of finite sequences of 0's and 1's, ordered by the relation "s is an initial segment of s'" (denoted by $s \leqq s'$). By a *tree* we mean any partial ordering isomorphic to $(S, \leqq)$. Thus a *subtree* of S is essentially any subset S' of S which has a single minimal element and is such that any element has exactly two immediate successors. A *chain* is an infinite linearly ordered subset of a tree. If T is a tree, the set of its chains will be denoted by $\mathscr{C}(T)$.

The present paper was motivated by the following problem raised by Brunel and Sucheston ([1]): Given a real Banach space E and a bounded sequence of elements of E indexed by S, say $(x_s)_{s\in S}$, is it possible to find a subtree $S' \subseteq S$ such that for any chain β of S':

(i) either the sequence $(x_s)_{s\in\beta}$ is a weak-Cauchy sequence,

(ii) or it is equivalent to the usual l^1-basis?

For the reader's convenience we recall the definitions of the notions involved:

0.1. Definition. Let E be a Banach space and let $(x_n)_{n\in N}$ be a sequence of elements in E.

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(i) (x_n) is a *weak-Cauchy sequence* if $\lim_{n\to\infty} f(x_n)$ exists for any element f in the dual space of E.

(ii) (x_n) is *equivalent to the usual l^1-basis* if there is a $\delta > 0$ such that for any integer n and any choice of scalars $\lambda_1, \cdots, \lambda_n$, the following holds:

$$\delta \sum_{i=1}^{n} |\lambda_i| \leqq \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|.$$

The answer to the above problem is positive.

Once we had proved it, we realized that it could be viewed as a consequence of the following Ramsey-type result.

0.2. THEOREM. *Let T be a tree and let $\mathscr{X}$ be an analytic set of chains; there exists a subtree T' of T such that*

(i) *either $\mathscr{C}(T') \subseteq \mathscr{X}$,*

(ii) *or $\mathscr{C}(T') \cap \mathscr{X} = \varnothing$.*

This theorem is a generalization of a theorem of Silver ([4]).

We now briefly describe the organization of the paper. §1 is devoted to the proof of Theorem 0.2 and, more generally, to the study of the following Ramsey property: if T is a tree, a set of chains $\mathscr{X}$ is said to be *Ramsey* if there exists a subtree $T' \subseteq T$ such that either $\mathscr{C}(T') \subseteq \mathscr{X}$ or $\mathscr{C}(T') \cap \mathscr{X} = \varnothing$. The proofs in this section use metamathematical techniques and we don't know how to avoid them.

In §2, we apply Theorem 0.2 to Banach space theory. Let $(x_s)_{s\in S}$ be a sequence of elements in a given Banach space and let $\mathscr{X}$ be the set of chains β of S such that:

(i) either the sequence $(x_s)_{s\in\beta}$ is a weak-Cauchy sequence,

(ii) or else it is equivalent to the usual l^1-basis.

A key step towards the solution of Brunel and Sucheston's problem is the following:

0.3. PROPOSITION. *$\mathscr{X}$ is coanalytic.*

(By coanalytic we mean the complement of an analytic set.)

We now recall a theorem of Rosenthal ([3]).

0.4. THEOREM (Rosenthal). *Let $(x_n)_{n\in\mathbf{N}}$ be a bounded sequence in a Banach space. Then (x_n) has a subsequence (x'_n) satisfying one of the two following alternatives:*

(i) *(x'_n) is a weak-Cauchy sequence,*

(ii) *(x'_n) is equivalent to the usual l^1-basis.*

The positive answer to the problem of Brunel and Sucheston is an easy consequence of Theorem 0.2, Theorem 0.4 and Proposition 0.3.

It should be emphasized that the present paper provides a new example where metamathematical ideas and techniques are used in order to solve a problem coming from a branch of "classical mathematics". Brunel and Sucheston needed the answer to their problem to achieve a probabilistic characterization of Banach spaces with separable dual ([1]). It is the author's conviction and hope that these examples will be more and more numerous.

§1. The Ramsey property for trees

In this section we use freely the definitions, notations and conventions of modern set theory, especially those which appear in Silver's paper ([4]) and we assume familiarity with this paper. For example, if X is a countable infinite set, $[X]^\omega$ denotes the set of all infinite subsets of X. Also, all topological notions concerning the space $P(X)$ are obtained by identifying $P(X)$ with 2^X and using the product of the discrete topology. $[X]^\omega$ is considered as a topological subspace.

The Ramsey property for trees has been defined in §0. We recall this definition as well as the definition of the usual Ramsey property.

1.1. Definition. (i) Let X be a set, $\mathscr{X}$ a subset of $[X]^\omega$; $\mathscr{X}$ is *Ramsey* if there is some infinite subset $Y \subseteq X$ such that either $[Y]^\omega \subseteq \mathscr{X}$ or $[Y]^\omega \cap \mathscr{X} = \emptyset$.

(ii) Let T be a tree, $\mathscr{X}$ a set of chains; $\mathscr{X}$ is *Ramsey* if there is a subtree $T' \subseteq T$ such that either $\mathscr{C}(T') \subseteq \mathscr{X}$ or $\mathscr{C}(T') \cap \mathscr{X} = \emptyset$. (Recall that $\mathscr{C}(T')$ is the set of chains of T'.)

The following lemma is the main tool which we will use to get results on the Ramsey property for trees.

1.2. Lemma. *Let T be a tree, $\mathscr{X}$ a subset of $\mathscr{C}(T)$; assume there exists a non-empty perfect closed set $P \subseteq \mathscr{C}(T)$ such that:*

(i) *for any $\zeta \in P\, [\zeta]^\omega \subseteq \mathscr{X}$,*

(ii) *for any pair of distinct elements of P, say ζ, ζ', their union $\zeta \cup \zeta'$ is not a chain,*

then there exists a subtree T' of T with

$$\mathscr{C}(T') \subseteq \mathscr{X}.$$

Proof. Without loss of generality, we assume $T = S$. Then, we define by a simultaneous induction on the length of the sequence s, two maps t, ξ:

$$t : S \to S,$$

$$\xi : S \to \mathscr{C}(S)$$

with the following properties:

(i) t is increasing,

(ii) $t(s\frown 0)$ and $t(s\frown 1)$ are incomparable,

(iii) $t(s) \in \xi(s) \in P$,

(iv) If $s \leqq s'$ then $\{u \leqq t(s) : u \in \xi(s)\} = \{u \leqq t(s) : u \in \xi(s')\}$.

In order to perform the construction, it is enough to show how $t(s\frown 0)$, $t(s\frown 1)$, $\xi(s\frown 0)$, $\xi(s\frown 1)$ are defined from $t(s)$, $\xi(s)$. Let X_s be

$$\{\gamma : \{u \leqq t(s) : u \in \xi(s)\} = \{u \leqq t(s) : u \in \gamma\}\}.$$

X_s is open (and closed) and $P \cap X_s$ is not empty because it contains $\xi(s)$; therefore, as P is a perfect set, one can choose two distinct elements in $P \cap X_s$, these two elements will be $\xi(s\frown 0)$, $\xi(s\frown 1)$. Now, by the hypothesis, $\xi(s\frown 0) \cup \xi(s\frown 1)$ is not a chain; it follows that one can pick $t(s\frown 0)$, $t(s\frown 1)$ such that

$$\begin{aligned}
&t(s\frown 0) \in \xi(s\frown 0),\\
&t(s\frown 1) \in \xi(s\frown 1),\\
&t(s\frown 0) \text{ and } t(s\frown 1) \text{ are incomparable},\\
&t(s\frown 0) \geqq t(s),\\
&t(s\frown 1) \geqq t(s).
\end{aligned}$$

Let S' be the image of t. Clearly S' is a subtree of S. Now let β be a chain of S'; there exists a chain γ of S such that $\beta = t(\gamma)$. The set $\{\xi(s) : s \in \gamma\}$ is a countable set of elements of P with a limit ζ. Clearly $\zeta \in P$ and $\beta \subseteq \zeta$. Therefore, as $[\zeta]^\omega \subseteq \mathscr{X}$, we get $\beta \in \mathscr{X}$.

Remark. Apparently, the axiom of choice has been used in the above proof. Actually it can be eliminated. For example, $\xi(s\frown 0)$ and $\xi(s\frown 1)$ can be chosen to be the minimal and maximal elements of $X_s \cap P$ (with respect to the usual ordering of P considered as homeomorphic to the Cantor space).

We now prove the following result:

1.3. Theorem. *Let $\mathscr{X}$ be a $\mathbf{\Sigma}^1_1$ set of chains of S; then $\mathscr{X}$ is Ramsey.*

This is a generalization of Silver's theorem (Any $\mathbf{\Sigma}^1_1$ subset of $[\omega]^\omega$ is Ramsey [4]).

Proof. First of all, we choose some canonical decomposition of $\mathscr{X}$ into $\aleph_1$ Borel sets; for example, if $\mathscr{X}$ is

$$\{\beta : \exists\alpha\psi(\alpha, \beta, \delta)\}$$

where ψ is an arithmetical formula and δ a fixed element of ${}^{\omega}\omega$, we let for $\theta < \aleph_1$

$$X_\theta = \{\beta : L_\theta[\beta, \gamma] \models \exists\alpha\psi(\alpha, \beta, \delta)\}.$$

Obviously

$$s\mathscr{X} = \bigcup_{\theta<\aleph_1} X_\theta.$$

We now fix a countable transitive model $\mathcal{N}$ of ZF_N such that δ is an element of $\mathcal{N}$ (where ZF_N is a large enough fragment of ZF to make our proof work). Let g be a Cohen-generic real over $\mathcal{N}$. We now work within the model $\mathcal{N}[g]$ (this is a model of $ZF_{N'}$, where N' can be made as large as we like by suitable choices of N). In this model, we apply Silver's theorem: let $\sigma(g)$ be the set of finite initial sequences of g; $\sigma(g)$ is a maximal chain in S and there exists an infinite $\gamma \subseteq \sigma(g)$ such that one of the following two alternatives holds:

(i) $[\gamma]^\omega \cap \mathscr{X} = \varnothing$,

(ii) $[\gamma]^\omega \subseteq \mathscr{X}$.

Actually, it is a corollary of the proof given by Silver ([4]) that (ii) is equivalent to the (apparently) stronger statement

(ii') $\exists\theta < \aleph_1\ [\gamma]^\omega \subseteq X_\theta$.

We now pick a forcing condition p and a term τ of the forcing language such that (in $\mathcal{N}$) one of the following holds (where Γ is a denotation for the generic real):

(*) $p \Vdash \tau$ is an infinite subset of $\sigma(\Gamma)$ and $[\tau]^\omega \cap \mathscr{X} = \varnothing$,

(**) for some ordinal $\theta < \aleph_1$,

$$p \Vdash \tau \text{ is an infinite subset of } \sigma(\Gamma) \text{ and } [\tau]^\omega \subseteq X_{\hat{\theta}}.$$

Let P be a perfect compact set of Cohen-generic reals over $\mathcal{N}$ extending the condition p. Let g be any element of P; if $\tau(g)$ is the interpretation of the term τ in $\mathcal{N}[g]$, then, clearly, $\tau(g)$ is an infinite subset of $\sigma(g)$. We let Π be the image of P under the continuous mapping $\tau : g \to \tau(g)$. Π is a compact set and given any pair of distinct elements of Π say $\tau(g)$, $\tau(g')$, $\tau(g) \cup \tau(g')$ is not a chain because $\tau(g)$ (resp. $\tau(g')$) is included in a unique maximal chain $\sigma(g)$ (resp. $\sigma(g')$). This argument also shows that τ is one-one and this implies that Π is a perfect set. So, in view of Lemma 1.2, it is enough to see that one of the following two statements holds:

(i) for any ζ in Π $[\zeta]^\omega \cap \mathscr{X} = \varnothing$,

(ii) for any ζ, in Π $[\zeta]^\omega \subseteq X_\theta$.

We now distinguish two cases:

Case 1. (*) holds. Then, if $\zeta = \tau(g)$ is in Π, $[\zeta]^\omega \cap \mathscr{X} = \varnothing$ is a Π_1^1 statement which holds in $\mathcal{N}(g)$. By Mostowski's absoluteness result, it remains true in the actual universe.

Case 2. (**) holds. Similarly, if $\zeta = \tau(g)$ is in Π, $[\zeta]^\omega \subseteq X_\theta$ can be coded as a Π_1^1 statement (about ζ, δ and any relation R_θ on ω having order type θ). As it holds in $\mathcal{N}[g]$, it remains true in the real world.

As usual, if one is ready to work in a stronger set theory than ZF, one can extend Theorem 1.3 to the $\mathbf{\Sigma}_2^1$ (P.C.A) sets.

1.4. THEOREM. *Assume* $\forall \alpha \aleph_1^{L[\alpha]} < \aleph_1$; *then any* $\mathbf{\Sigma}_2^1$ *set of chains in S is Ramsey.*

This theorem shows, inter alia, that the existence of measurable cardinals implies that $\mathbf{\Sigma}_2^1$ sets of chains are Ramsey. It extends an analog result of Silver for $\mathbf{\Sigma}_2^1$ subsets of $[\omega]^\omega$.

PROOF OF THE THEOREM. We first choose some canonical decomposition of $\mathscr{X}$ into $\aleph_1$ Π_1^1 sets.

For example, if $\mathscr{X}$ is

$$\{\beta : \exists \alpha \varphi(\alpha, \beta, \delta)\}$$

where φ is a Π_1^1 formula and δ a fixed element of ${}^\omega\omega$, we let for $\theta < \aleph_1$

$$X_\theta = \{\beta : \exists \alpha \in L_\theta[\beta, \delta]\, \varphi(\alpha, \beta, \delta)\}.$$

Clearly $\mathscr{X} = \bigcup_{\theta < \aleph_1} X_\theta$.

Then, we consider a Cohen generic extension $L[\delta][g]$ of the model $L[\delta]$. The following lemma can be extracted from the proof of Theorem 2 in Silver's paper [4]:

1.5. LEMMA. *In* $L[\delta][g]$ *one can find a notion of forcing* $\mathbf{P}$ *with cardinality* $\leqq \aleph_2$ *such that in the corresponding generic extension* $\mathcal{M}$ *the cardinals are preserved and there is an infinite subset* γ *of* g *such that one of the following holds*:

(i) $[\gamma]^\omega \cap \mathscr{X} = \varnothing$,

(ii) $\exists \theta < \aleph_1 \quad [\gamma]^\omega \subseteq X_\theta$.

We now consider $\mathcal{M}$ as a generic extension of $L[\delta]$ via a notion of forcing $\mathbf{P}'$. From the lemma we can see that there exists a forcing condition $p \in \mathbf{P}'$ and two terms τ_1, τ_2 of the forcing language such that $p \Vdash \tau_1$ is infinite $\& \tau_2 \subseteq \tau_1 \& \tau_1$ is a maximal branch $\& \tau_1 \notin L[\hat{\delta}]$ and

(*) either $p \Vdash [\tau_2]^\omega \cap \mathscr{X} = \varnothing$,

(**) or for some $\theta < \aleph_1^{L[\delta]}$ $p \Vdash [\tau_2]^\omega \subseteq X_\theta$.

Following section III 1.10 of Solovay's paper [5], we consider a map $\pi : S \to \mathbf{P}'$ which has the following properties:

(i) π is increasing,

(ii) $\pi(\emptyset) = p$,

(iii) given any maximal branch β of S, there is a unique $\mathbf{P}'$-generic set G_β containing $\{\pi(s) : s \in \beta\}$,

(iv) furthermore, $\beta \neq \beta'$ implies that $G_\beta \times G_{\beta'}$ is $\mathbf{P}' \times \mathbf{P}'$-generic over $L[\delta]$.

The map $\lambda : \beta \to \tau_2(G_\beta)$ is continuous. Also, it is one-one. To see this let $\beta \neq \beta'$; $\tau_2(G_\beta) = \tau_2(G_{\beta'})$ would imply $\tau_1(G_\beta) = \tau_1(G_{\beta'})$ (because $\tau_1(G_\beta)$ and $\tau_1(G_{\beta'})$ are maximal branches). Now, because $G_\beta \times G_{\beta'}$ is $\mathbf{P}' \times \mathbf{P}'$-generic over $L[\delta]$, this implies $\tau_1(G_\beta) \in L[\delta]$; contradiction. As the domain of λ is a compact perfect set, λ is an homeomorphism, therefore its image is also a perfect set. Also, the argument used to show that λ is one-one actually shows that $\beta \neq \beta'$ implies $\lambda(\beta) \cup \lambda(\beta')$ is not a chain. Therefore, in order to apply Lemma 1.2, it is enough to see that one of the following statements holds:

(i) for any β $[\lambda(\beta)]^\omega \cap \mathscr{X} = \emptyset$,

(ii) for any β $[\lambda(\beta)]^\omega \subseteq X_\theta$.

We distinguish two cases.

Case 1. (*) holds; then the statement $[\lambda(\beta)]^\omega \cap \mathscr{X} = \emptyset$ is a Π_2^1 statement which is known to hold in $L[\delta][G_\beta]$; by Shoenfield absoluteness lemma it holds in the actual universe.

Case 2. (**) holds; a similar argument works.

Before we close §1, let us mention the following result which shows the consistency of the theory ZF + DC + Any set of chains is Ramsey (provided ZF + there exists an inaccessible cardinal is consistent).

1.6. THEOREM. *In Solovay's model (see [5]), any set of chains is Ramsey.*

The proof is very similar to the proof of Theorem 1.5, using the fact that all subsets of $[\omega]^\omega$ are Ramsey in Solovay's model; this fact is due to Mathias ([2]).

§2. An extension of a theorem of Rosenthal

The following result is an extension of Rosenthal's theorem (quoted in the introduction as Theorem 0.4).

2.1. THEOREM. *Let E be a real Banach space and let $(x_s)_{s\in S}$ be a bounded sequence of elements of E; there exists a subtree $S' \subseteq S$ such that one of the following holds:*

(i) *For any chain β of S' $(x_s)_{s\in\beta}$ is a weak-Cauchy sequence.*

(ii) *For any chain β of S' $(x_s)_{s\in\beta}$ is equivalent to the usual l^1-basis.*

In order to prove Theorem 2.1 we need the following.

2.2. PROPOSITION. *Let $(x_s)_{s\in S}$ be a sequence of elements in a Banach space E; then*

(i) *$\{\beta : \beta$ is a chain and $(x_s)_{s\in\beta}$ is a weak Cauchy sequence$\}$ is coanalytic,*

(ii) *$\{\beta : \beta$ is a chain and $(x_s)_{s\in\beta}$ is equivalent to the usual l^1 basis$\}$ is the intersection of a G_δ with an F_σ.*

The proof of Proposition 2.2 will be given in detail, in order to be accessible to the reader who is not a specialist in descriptive set theory.

Recall that $P(S)$ is endowed with a topology whose basic open sets are the sets

$$U(s_1,\cdots,s_n;s'_1,\cdots,s'_k)=\{\alpha : s_1\in\alpha,\cdots,s_n\in\alpha, s'_1\notin\alpha,\cdots,s'_k\notin\alpha\}$$

when $(s_1,\cdots,s_n)$, $(s'_1,\cdots,s'_k)$ vary over the finite sequences of elements of S.

We now consider the space $X=P(S)\times\mathbf{R}^S$, where $\mathbf{R}^S$ is endowed with the product of the usual topology.

We assume $(x_s)_{s\in S}$ is a fixed sequence.

2.3. LEMMA. *Let X_1 be the set of pairs (α,f) in X such that there exists a linear functional $\bar f$ of norm $\leqq 1$ with $\bar f(x_s)=f(s)$ for any s; X_1 is a closed set.*

PROOF. A linear functional $\bar f$ of norm $\leqq 1$ with $\bar f(x_s)=f(s)$ for any $s\in S$ exists if and only if the following inequalities hold:

$$|\alpha_1 f(s_1)+\cdots+\alpha_k f(s_k)|\leqq\|\alpha_1 x_{s_1}+\cdots+\alpha_k x_{s_k}\|$$

for any sequence of scalars $(\alpha_1,\cdots,\alpha_k)$ and any sequence of elements of S $(s_1,\cdots,s_k)$. Each inequality defines a closed set. X_1, the intersection of all these closed sets, is closed.

2.4. LEMMA. *Let X_2 be the set of pairs (α,f) in X such that the sequence $(f(s))_{s\in\alpha}$ is not Cauchy; X_2 is $G_{\delta\sigma}$.*

PROOF. X_2 has the following definition $(\alpha,f)\in X_2$ iff $\exists q\in\mathbf{Q}$ $q>0$ and $\forall\sigma$ (σ is a finite set of elements of $S\Rightarrow\exists s\,\exists s'$ ($s\in\alpha$ and $s'\in\alpha$ and $|f(s)-f(s')|>q$ and $s\notin\sigma$ and $s'\notin\sigma$)).

Notice that for fixed q and σ

$$A_{q,\sigma}=\{(\alpha,f):\exists s\,\exists s'(s\in\alpha\text{ and }s'\in\alpha\text{ and }|f(s)-f(s')|>q$$
$$\text{and }s\notin\sigma\text{ and }s'\notin\sigma)\}$$

is an open set in X. Now

$$X_2 = \bigcup_{\substack{q \in Q \\ q > 0}} \bigcap_{\substack{\sigma \subseteq S \\ \sigma \text{ finite}}} A_{q,\sigma}.$$

2.5. LEMMA. *The set of all chains of S is a G_δ.*

PROOF. It is enough to write down the following definition of the set of all chains $\mathscr{C}(S)$:

$$\beta \in \mathscr{C}(S) \text{ iff } \forall s\, \forall s' (s \in \beta \text{ and } s' \in \beta \rightarrow s \text{ and } s' \text{ are comparable})$$

$$\text{and } \forall s \in \beta\, \exists s' \in \beta\, (s' \neq s \text{ and } s' \geqq s).$$

We are now able to give the proof of Proposition 2.2(i). As $\mathscr{C}(S)$ is a G_δ it is enough to show that

$$Y = \{\beta : (x_s)_{s \in \beta} \text{ is not a weak Cauchy sequence}\}$$

is analytic. But a given β is in Y if and only if there exists a linear functional $\bar{f}$ of norm $\leqq 1$ such that $\bar{f}(x_s)_{s \in \beta}$ is not Cauchy. This means exactly that β belongs to the projection on $P(S)$ of the Borel set $X_1 \cap X_2$.

To prove Proposition 2.2(ii) we consider for $\delta > 0$ the set Z_δ of β which satisfy the inequalities

$$\delta \sum_{i=1}^{n} |\alpha_i| \leqq \left\| \sum_{i=1}^{n} \alpha_i x_{s_i} \right\|$$

for all sequences of scalars $\alpha_1, \cdots, \alpha_n$ and all sequences of elements $s_1, \cdots, s_n$ in β; this set Z_δ is closed. Now

$$\{\beta : \beta \text{ is a chain and } (x_s)_{s \in \beta} \text{ is equivalent to the usual } l^1 \text{ basis}\}$$

is the intersection of the G_δ set $\mathscr{C}(S)$ with the F_σ set $\bigcup_{q \in Q}^{q > 0} Z_q$.

We now complete the proof of Theorem 2.1.

By applying twice Theorem 1.3, one can find a subtree $S' \subseteq S$ such that one of the following three possibilities holds:

(i) for any chain β of S' $(x_s)_{s \in \beta}$ is a weak-Cauchy sequence,

(ii) for any chain β of S' $(x_s)_{s \in \beta}$ is equivalent to the usual l^1 basis,

(iii) for any chain β of S' $(x_s)_{s \in \beta}$ is not a weak-Cauchy sequence; neither is it equivalent to the usual l^1 basis.

Thus, what we need is to exclude the third possibility. Assume it happens and apply Rosenthal's theorem (0.4) to any chain β of S': there exists an infinite $\beta' \subseteq \beta$ such that

either $(x_s)_{s \in \beta'}$ is a weak Cauchy sequence
or else it is equivalent to the usual l^1-basis.

But β' is a chain of S'; contradiction.

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